Partial wave analysis in K-matrix formalism

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Abstract

A description is given of the K-matrix formalism. The formalism, which is normally applied to two-body scattering processes, is generalized to production of two-body channels with final-state interactions. A multi-channel treatment of production of resonances has been worked out in the P-vector approach of Aitchison. An alternative approach, derived from the P-vector, gives the production amplitude as a product of the T-matrix for a two-body system and a vector Q specifying its production. This formulation, called Q-vector approach here, has also been worked out. Examples of practical importance are given.

1 Introduction

The K-matrix formalism provides an elegant way of expressing the unitarity of the S-matrix for the processes of the type $ab \to cd$. It has been originally introduced by Wigner[1] and Wigner and Eisenbud[2] for study of resonances in nuclear reactions. The first use in particle physics goes back to an analysis of resonance production in Kp scattering by Dalitz and Tuan [3]. A comprehensive review is found in [4]. In this paper we give a concise description of the K-matrix formalism for ease of reference. Its generalizations to arbitrary production processes are covered in some detail.

The reader is referred to the text book by Martin and Spearman [5] for some of the material covered in this note. However, one must note that the definitions given in this paper are different from those used by Martin and Spearman. Cahn and Landshoff [6] and Au, Morgan and Pennington [7] have used the same definitions as those adopted by us.

The unitary relationship involves a bilinear product, and one must exercise care with constant factors, as there is essentially no freedom with the coefficients. The derivation for the cross section from unitarity follows a well-defined prescription and, once defined, one must again adhere to it rigorously.

This paper is intended to be useful for experimental physicists working on meson spectroscopy. For this purpose a reasonably self-contained and elementary account has been made of the K-matrix formalism appropriate for a study of overlapping and many channel resonances.

2 S-Matrix and Unitarity

Consider a two-body scattering of the type $ab \to cd$. The differential cross section is given in terms of the invariant amplitude $\mathcal{M}$ and the 'scattering amplitude' f through

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{1}{(8\pi)^2\, s}\left(\frac{q_f}{q_i}\right)|\mathcal{M}_{fi}|^2 = |f_{fi}(\Omega)|^2 \tag{1}$$

where 'i' and 'f' stand for the initial and final states; $\Omega = (\theta, \phi)$ denotes the usual spherical coordinate system; and $s = m^2$ is the square of the center–of–mass (CM) energy. The $q_i(q_f)$ is the breakup momentum in the initial(final) system. [The observed cross section is in reality the average of the initial spin states and the sum over all final spin states— this is suppressed here for simplicity.] The scattering amplitude can be expanded in terms of the partial-wave amplitudes

$$f_{fi}(\Omega) = \frac{1}{q_i} \sum_J (2J+1) T^J_{fi}(s) D^{J*}_{\lambda\mu}(\phi, \theta, 0) \tag{2}$$

where $\lambda = \lambda_a - \lambda_b$ and $\mu = \lambda_c - \lambda_d$ in terms of the helicities of the particles involved in the scattering $ab \to cd$. Note that this 'scattering amplitude' is a factor of two bigger than that with a more common definition (for example, see Section 5.1, Chung [8]). One may in addition note that the argument of the D-function is frequently given as $(\phi, \theta, -\phi)$ (see Jacob and Wick [9] and Martin and Spearman [5]). Integrating the differential cross section over the angles, one finds, for the cross section in the partial wave J,

$$\sigma^J_{fi} = \left(\frac{4\pi}{q_i^2}\right)(2J+1)|T^J_{fi}(s)|^2 \tag{3}$$

Note that T^J has no unit; the unit for the cross section is being carried by q_i^2. It is necessary to define more precisely the initial and the final states

$$\begin{aligned} |i\rangle &= |ab, JM\lambda_a\lambda_b\rangle \\ |f\rangle &= |cd, JM\lambda_c\lambda_d\rangle \end{aligned} \tag{4}$$

where M is the z-component of total spin J in a coordinate system fixed in the overall CM frame and the notations $\{ab\}$ and $\{cd\}$ designate additional informations needed to fully specify the initial and the final states. Because of conservation of angular momentum, an initial state in $|JM\rangle$ remains the same in the scattering process. Note the normalization (see Section 4.2, Chung [8])

$$\langle f|i\rangle = \delta_{ij} \tag{5}$$

In the remainder of this section and in subsequent sections, it is be understood that the ket states mentioned always refer to those of (4). In particular, explicit references to the total angular momentum J will be suppressed. Note that, with this convention, one has eliminated the necessity of specifying continuum variables such as angles and momenta.

In general, the amplitude that an initial state $|i\rangle$ will be found in the final state $|f\rangle$ is

$$S_{fi} = \langle f|S|i\rangle \tag{6}$$

where S is called the scattering operator. One may remove the probability that the initial and final states do not interact at all, by defining the transition operator T through

$$S = I + 2i\ T \tag{7}$$

where I is the identity operator. The factors 2 and i have been introduced for convenience. From conservation of probability, one deduces that the scattering operator S is unitary, i.e.

$$S\ S^\dagger = S^\dagger S = I \tag{8}$$

From the unitarity of the S, one gets

$$T - T^\dagger = 2i\ T^\dagger T = 2i\ TT^\dagger \tag{9}$$

In terms of the inverse operators, (9) can be rewritten

$$(T^\dagger)^{-1} - T^{-1} = 2iI \tag{10}$$

One may further transform this expression into

$$(T^{-1} + iI)^\dagger = T^{-1} + iI \tag{11}$$

One is now ready to introduce the K operator via

$$K^{-1} = T^{-1} + iI \tag{12}$$

From (11) one finds that the K operator is Hermitian, i.e.

$$K = K^\dagger \tag{13}$$

From time reversal invariance of S and T it follows, that the K operator must be symmetric, i.e. the K-matrix may be chosen to be real and symmetric.

One can eliminate the inverse operators in (12) by multiplying by K and T from left and right and vice versa, to obtain

$$T = K + iTK = K + iKT \tag{14}$$

This shows that K and T operators commute, i.e.

$$[K, T] = 0 \tag{15}$$

and that, solving for T, one gets

$$T = K(I - iK)^{-1} = (I - iK)^{-1}K \tag{16}$$

and

$$S = (I + iK)(I - iK)^{-1} = (I - iK)^{-1}(I + iK) \tag{17}$$

Note that the T-matrix is complex only through the i that appears in this formula, i.e. T^{-1} has been explicitly broken up into its real and imaginary parts, see equation (12).

It is also useful to split T into its real and imaginary part. From (14) one finds, noting that K is a real matrix,

$$\begin{aligned} Re\ T &= (I + K^2)^{-1}K = K(I + K^2)^{-1} \\ Im\ T &= (I + K^2)^{-1}K^2 = K^2(I + K^2)^{-1} \end{aligned} \tag{18}$$

Combining this result with (9), one finds that the unitarity takes on the simple form

$$Im\ T = T^*T = TT^* \tag{19}$$

Or, from (12), one gets

$$Im\ T^{-1} = -I \tag{20}$$

Consider now an isoscalar $\pi\pi$ scattering in S-wave below $\sqrt{s} = 1$GeV. This is a single-channel problem and unitarity is rigorously maintained. From (8), one may set

$$S = e^{2i\delta} \tag{21}$$

where δ is the familiar phase shift. The transition amplitude T is given, from (7),

$$T = e^{i\delta} \sin\delta \tag{22}$$

Note that the factors 2 and i in (7) make the T attain the simple, familiar form. This formula shows that the trajectory of T in the complex plane (Argand diagram) is a circle of a unit diameter with its center at $(0, i/2)$. This is the so-called unitarity circle and the physically allowed T should remain at or within this circle. The S-wave cross section is, from (3),

$$\sigma = \left(\frac{4\pi}{q_i^2}\right) \sin^2\delta \tag{23}$$

The K-matrix for this case is simply

$$K = \tan\delta \tag{24}$$

A pole in K is therefore associated with $\delta = \pi/2$.

Consider next a two-channel problem in which the S-matrix may be expressed as 2×2 matrices. Let S_{ij} be symmetric; then it would take six parameters to represent three complex variables: S_{11}, S_{12}, and S_{22}. The unitarity relationship

$$S_{ik} S_{jk}^* = \delta_{ij} \tag{25}$$

imposes three independent equations. This shows that the S-matrix depends on just three parameters. It can be shown readily that the matrix elements are

$$\begin{aligned} S_{11} &= \eta\ e^{2i\delta_1} \\ S_{22} &= \eta\ e^{2i\delta_2} \\ S_{12} &= i\sqrt{1-\eta^2}\ e^{i\Phi_{12}}, \quad \Phi_{12} = \delta_1 + \delta_2 \end{aligned} \tag{26}$$

where δ_i is the phase shift for channel i and η is the inelasticity ($\eta \leq 1$). Note that there exists only one inelasticity in the two-channel case.

Turning to the K-matrix representation of the T-matrix, let

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \tag{27}$$

where $K_{12} = K_{21}$ and K_{ij} = real. Then, from (12), one finds

$$T = \frac{1}{1 - D - i(K_{11} + K_{22})} \begin{pmatrix} K_{11} - iD & K_{12} \\ K_{21} & K_{22} - iD \end{pmatrix} \tag{28}$$

where

$$D = K_{11}K_{22} - K_{12}^2 \tag{29}$$

3 Lorentz-Invariant T-Matrix

The transition amplitudes T as defined in (7) is not Lorentz invariant. The invariant amplitude is defined through two-body wave functions for the initial and the final state, and the process of the derivation involves proper normalizations for the two-particle states (see Section 5.1, Chung [8]). The resulting invariant amplitude contains the inverse square-root of the two-body phase space elements in the initial and the final states. The Lorentz-invariant amplitude, denoted $\widehat{T}$, is thus given by

$$T_{ij} = \{\rho_i\}^{\frac{1}{2}} \ \widehat{T}_{ij} \ \{\rho_j\}^{\frac{1}{2}} \tag{30}$$

In matrix notation, one may write

$$T = \{\rho\}^{\frac{1}{2}} \ \widehat{T} \ \{\rho\}^{\frac{1}{2}} \tag{31}$$

and

$$S = I + 2i \ \{\rho\}^{\frac{1}{2}} \ \widehat{T} \ \{\rho\}^{\frac{1}{2}} \tag{32}$$

where the phase-space 'matrix' is diagonal by definition, i.e.

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \tag{33}$$

and

$$\rho_1 = \frac{2q_1}{m} \quad \text{and} \quad \rho_2 = \frac{2q_2}{m} \tag{34}$$

The q_i is the breakup momentum in channel i. (Here one considers a two-channel problem for simplicity without loss of generality.) The unitarity demands that, from (19) and (20),

$$Im \ \widehat{T} = \widehat{T}^* \rho \widehat{T} = \widehat{T} \rho \widehat{T}^* \tag{35}$$

and

$$Im\ \widehat{T}^{-1} = -\rho \tag{36}$$

It is in this form one encounters most frequently the unitary conditions of the transition matrix in the literature.

The cross section in the Jth partial wave is given by, from (3),

$$\sigma^J_{fi} = \left(\frac{16\pi}{s}\right)\left(\frac{\rho_f}{\rho_i}\right)(2J+1)|\widehat{T}^J_{fi}(s)|^2 \tag{37}$$

Note that this formula embodies the familiar presence of the flux factor of the initial system and the phase-space factor of the final system in the process $ab \to cd$. In the K-matrix formalism, one allows for ρ to become imaginary below a given threshold; however, the cross section above has no meaning below a threshold, and one could then modify the expression above by multiplying it with two step functions: $\theta(\rho_i^2)$ and $\theta(\rho_f^2)$.

One may recapitulate the expressions for the differential cross section and its partial-wave expansion in terms of the invariant amplitudes $\widehat{T}^J_{fi}(s)$. For the purpose, one defines the 'invariant scattering amplitude'

$$\widehat{T}_{fi}(\Omega) = \sum_J (2J+1)\widehat{T}^J_{fi}(s) D^{J\,*}_{\lambda\mu}(\phi,\theta,0) \tag{38}$$

and the differential cross section is given by

$$\frac{d\sigma_{fi}}{d\Omega} = \left(\frac{4}{s}\right)\left(\frac{\rho_f}{\rho_i}\right)|\widehat{T}_{fi}(\Omega)|^2 \tag{39}$$

The initial and final density of states are, with $s = m^2$,

$$\begin{aligned}
\rho_i &= \sqrt{\left[1-\left(\frac{m_a+m_b}{m}\right)^2\right]\left[1-\left(\frac{m_a-m_b}{m}\right)^2\right]} \\
\rho_f &= \sqrt{\left[1-\left(\frac{m_c+m_d}{m}\right)^2\right]\left[1-\left(\frac{m_c-m_d}{m}\right)^2\right]}
\end{aligned} \tag{40}$$

in terms of the particle masses involved in the scattering $ab \to cd$. Note that these phase-space factors are normalized such that

$$\rho_i \to 1 \quad \text{as} \quad m^2 \to \infty \tag{41}$$

The invariant amplitude $\widehat{T}_{fi}(\Omega)$ is unitless, and has a partial-wave expansion (38). The partial-wave amplitude $\widehat{T}^J_{fi}(s)$ is related to the $\widehat{K}$-matrix via (45), and unitarity is preserved if the $\widehat{K}$-matrix is taken to be real and symmetric. It should be noted that the formula for the differential cross section (39) has no 'arbitrary' numerical factors. The 'conventional' invariant amplitude, introduced in (1), is given by

$$\mathcal{M}_{fi} = 16\pi\, \widehat{T}_{fi}(\Omega) \tag{42}$$

One may consider again the isoscalar $\pi\pi$ scattering in S-wave below 1.0 GeV. In terms of the phase-shift δ, the invariant amplitude is given by, from (22),

$$\widehat{T} = \frac{1}{\rho} e^{i\delta} \sin\delta \tag{43}$$

and when substituted into (37) the cross section (23) results. These expressions are very familiar, and they demonstrate clearly the interplay between the phase shifts, the invariant amplitudes and the cross sections.

One can similarly define the invariant analogue of the K-matrix through

$$K = \{\rho\}^{\frac{1}{2}}\ \widehat{K}\ \{\rho\}^{\frac{1}{2}} \tag{44}$$

From (12) one sees that

$$\widehat{K}^{-1} = \widehat{T}^{-1} + i\rho \tag{45}$$

which leads to

$$\widehat{T} = \widehat{K} + i\widehat{K}\rho\widehat{T} = \widehat{K} + i\widehat{T}\rho\widehat{K} \tag{46}$$

and

$$\widehat{T}\rho\widehat{K} = \widehat{K}\rho\widehat{T} \tag{47}$$

Solving for $\widehat{T}$, one obtains

$$\widehat{T} = \widehat{K}(I - i\rho\widehat{K})^{-1} = (I - i\widehat{K}\rho)^{-1}\widehat{K} \tag{48}$$

and, from (17),

$$\begin{aligned} S &= (I + i\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})(I - i\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})^{-1} \\ &= (I - i\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}})^{-1}(I + i\{\rho\}^{\frac{1}{2}}\ \widehat{K}\{\rho\}^{\frac{1}{2}}) \end{aligned} \tag{49}$$

Note that $\widehat{K}$ and ρ do not commute. The Lorentz-invariant T-matrix is then given by

$$\widehat{T} = \frac{1}{1-\rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \begin{pmatrix} \widehat{K}_{11} - i\rho_2\widehat{D} & \widehat{K}_{12} \\ \widehat{K}_{21} & \widehat{K}_{22} - i\rho_1\widehat{D} \end{pmatrix} \tag{50}$$

where

$$\widehat{D} = \widehat{K}_{11}\widehat{K}_{22} - \widehat{K}_{12}^2 \tag{51}$$

4 Momentum-Space Representation of S-Matrix

It is possible to express the S-matrix elements directly as ratios of certain functions in momentum space. This technique is sometimes referred to as the Jost function representation[10] of the S-matrix. We follow here the derivation as given by Kato[11].

For the sake of simplicity we deal here only with a 2×2 S-matrix. Practical and important examples are the coupled channel problems dealing with the isoscalar S-wave $\pi\pi$ and $K\bar{K}$ states and the isovector S-wave $\pi\eta$ and $K\bar{K}$ states. Let q_1 and q_2 be the momenta for the two channels under consideration. The transition matrix $\widehat{T}$ should be a hermitean analytic function, i.e. in the energy variable s [5]

$$\widehat{T}^*(s) = \widehat{T}(s^*) \tag{52}$$

but, when expressed as a function of q_i, one must have

$$\widehat{T}^*(q_1, q_2) = \widehat{T}(-q_1^*, -q_2^*) \tag{53}$$

in order to preserve the unitary condition (36). It is convenient to introduce a new notation $A(q_1, q_2) = \widehat{T}^{-1}$. Evidently, $A(q_1, q_2)$ obeys the same relationship (53), so that one may write, using (45),

$$A(-q_1^*, -q_2^*) = A(q_1, q_2) + 2i\rho \tag{54}$$

This is the fundamental equation, from which various special formulae can be derived and the results analytically continued. Remember that there is only one variable s in the problem, and analytic continuation implies that one is dealing with a study of structures in the complex s-plane. The unitarity condition imposes branch points, and different paths through the cuts give rise to different Riemann sheets (see the section on the Flatté formula for an example). The transformations $q_i \to -q_i$ where $i = 1, 2$ or both, introduced in the following, are in fact those that generate four Riemann sheets in the two-channel problem treated here.

In the physical region in which both q_1 and q_2 are real and positive, it is convenient to rewrite the unitarity condition as follows:

$$A(-q_1, -q_2) = A(q_1, q_2) + 2i\rho \tag{55}$$

Consider, for the moment, a physical region for s below Channel 2 but above Channel 1, i.e. q_1 is real but q_2 is purely imaginary. In this case there is only one open channel, namely Channel 1. More generally one must imagine that each ρ_i is accompanied by a step function at its threshold i. One then finds, from (54),

$$A(-q_1, q_2) = A(q_1, q_2) + 2i\rho^{[1]} \tag{56}$$

where the superscript [1]([2]) signifies that the diagonal element 22(11) is zero. By analytic continuation this relationship should be true also in the physical region for s above Channel 2. Now let $q_1 \to -q_1$ in (55) and add the resulting equation to (56), to obtain

$$A(q_1, -q_2) = A(q_1, q_2) + 2i\rho^{[2]} \tag{57}$$

If we put $\widehat{T} = ND^{-1}$ or $A = DN^{-1}$ where D depends on q_i while N does not, then one gets, from (55),

$$D(-q_1, -q_2) = D(q_1, q_2) + 2i\rho N \tag{58}$$

and, from (56) and (57),

$$\begin{aligned} D(-q_1, q_2) &= D(q_1, q_2) + 2i\rho^{[1]} N \\ D(q_1, -q_2) &= D(q_1, q_2) + 2i\rho^{[2]} N \end{aligned} \tag{59}$$

Note that

$$\{\rho\}^{\frac{1}{2}}\ S\ \{\rho\}^{-\frac{1}{2}} = I + 2i\rho\widehat{T} = (D + 2i\rho N)D^{-1} \tag{60}$$

so that one finds

$$\det\ S = \det\ \{(D + 2i\rho N)D^{-1}\} = \det\ D(-q_1, -q_2)/\det\ D(q_1, q_2) \tag{61}$$

and

$$\begin{aligned} S_{11} &= \det \{(D + 2i\rho^{[1]}N)D^{-1}\} = \det\ D(-q_1, q_2)/\det\ D(q_1, q_2) \\ S_{22} &= \det \{(D + 2i\rho^{[2]}N)D^{-1}\} = \det\ D(q_1, -q_2)/\det\ D(q_1, q_2) \end{aligned} \tag{62}$$

One may define a function d through

$$d(q_1, q_2) = \det\ D(q_1, q_2) \tag{63}$$

which is hermitean analytic except for the branch points at q_i and satisfies

$$d^*(q_1, q_2) = d(-q_1^*, -q_2^*) \tag{64}$$

In terms of this function the elements of the S-matrix are given by

$$\begin{aligned} S_{11} &= \frac{d(-q_1, q_2)}{d(q_1, q_2)} \\ S_{22} &= \frac{d(q_1, -q_2)}{d(q_1, q_2)} \\ S_{11}S_{22} - S_{12}^2 &= \frac{d(-q_1, -q_2)}{d(q_1, q_2)} \end{aligned} \tag{65}$$

One notes that the singularity structure of the S-matrix is clearly delineated in this approach: the poles of the S-matrix derive from the zeroes in $d(q_1, q_2)$. In fact Morgan and Pennington[12] used this form to explore the nature of the $f_0(975)$.

Martin et al.[13] point out that a requirement of the unitarity condition (25) in the physical region is that the following inequality holds:

$$|d(-q_1, q_2)| \leq |d(q_1, q_2)| \tag{66}$$

This leads to the following explicit expression for the off-diagonal S-matrix element:

$$S_{12} = \frac{i\sqrt{|d(q_1, q_2)|^2 - |d(q_1, -q_2)|^2}}{[d(q_1, q_2)]_+} \tag{67}$$

where $[a]_+ = [-a]_+$. One has adopted here a definition such that $-iS_{12}$ is real and positive if $d(q_1, q_2)$ is real. One can easily show that the unitarity conditions (25) are satisfied in the physical region. It is instructive to express $A = \widehat{T}^{-1}$ directly in terms of $d(q_1, q_2)$. From (32), (65) and (67) one finds

$$\begin{aligned} A_{11} &= \frac{2i\rho_1[d(q_1,-q_2) - d(q_1,q_2)]}{d(q_1,q_2) + d(-q_1,-q_2) - d(q_1,-q_2) - d(-q_1,q_2)} \\ A_{22} &= \frac{2i\rho_2[d(-q_1,q_2) - d(q_1,q_2)]}{d(q_1,q_2) + d(-q_1,-q_2) - d(q_1,-q_2) - d(-q_1,q_2)} \\ A_{12} &= \frac{2\sqrt{\rho_1\rho_2\{|d(q_1,q_2)|^2 - |d(q_1,-q_2)|^2\}}}{[d(q_1,q_2) + d(-q_1,-q_2) - d(q_1,-q_2) - d(-q_1,q_2)]_+} \end{aligned} \tag{68}$$

Note that the denominators are real and A_{12} is real and positive in the physical region. It is seen that these satisfy the properties (55), (56) and (57).

The function $d(q_1, q_2)$ can be derived from (48) and (49) in terms of $\widehat{K}$:

$$d(q_1,q_2) = \phi(s)[1 - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22}) - \rho_1\rho_2 \ \det\widehat{K}] \tag{69}$$

where $\phi(s)$ and $\widehat{K}$ are real functions of s. Equivalently, they are real functions of q_1^2 or q_2^2. Note that this formula satisfies the condition (64). Another simple example of the function $d(q_1, q_2)$ is given later in the section on the Flatté formula.

5 Resonances in K-matrix Formalism

Resonances should appear as a sum of poles in the K-matrix. In the approximation of resonance domination for the amplitudes, one has therefore

$$K_{ij} = \sum_\alpha \frac{g_{\alpha i}(m) g_{\alpha j}(m)}{m_\alpha^2 - m^2} \tag{70}$$

and

$$\widehat{K}_{ij} = \sum_\alpha \frac{g_{\alpha i}(m) g_{\alpha j}(m)}{(m_\alpha^2 - m^2)\sqrt{\rho_i\rho_j}} \tag{71}$$

where the sum on α goes over the number of resonances with masses m_α, and the residue functions (expressed in units of energy) are given by

$$g_{\alpha i}^2(m) = m_\alpha \Gamma_{\alpha i}(m) \tag{72}$$

where $g_{\alpha i}(m)$ is real (but it could be negative) above the threshold for channel i. The width $\Gamma_\alpha(m)$ is

$$\Gamma_\alpha(m) = \sum_i \Gamma_{\alpha i}(m) \tag{73}$$

for each resonance α.

Consider now a resonance α coupling to n open two-body channels, i.e. the mass m_α is above the threshold of all the two-body channels. The partial widths may be given an expression

$$\Gamma_{\alpha i}(m) = \frac{g^2_{\alpha i}(m)}{m_\alpha} = \gamma^2_{\alpha i}\Gamma^0_\alpha \left[B^l_{\alpha i}(q, q_\alpha)\right]^2 \rho_i \tag{74}$$

and the residue function by

$$g_{\alpha i}(m) = \gamma_{\alpha i}\sqrt{m_\alpha \Gamma^0_\alpha}\; B^l_{\alpha i}(q, q_\alpha)\sqrt{\rho_i} \tag{75}$$

The $B(m)$'s are ratios of the usual Blatt-Weisskopf barrier factors [14] in terms of the breakup momentum in channel i and the resonance breakup momentum q_α for the orbital angular momentum l,

$$B^l_{\alpha i}(q, q_\alpha) = \frac{F_l(q)}{F_l(q_\alpha)} \tag{76}$$

where, with the normalization $F_l(1) = 1$, one has

$$\begin{aligned}
F_0(q) &= 1 \\
F_1(q) &= \sqrt{\frac{2z}{z+1}} \\
F_2(q) &= \sqrt{\frac{13z^2}{(z-3)^2 + 9z}} \\
F_3(q) &= \sqrt{\frac{277z^3}{z(z-15)^2 + 9(2z-5)}} \\
F_4(q) &= \sqrt{\frac{12746z^4}{(z^2-45z+105)^2 + 25z(2z-21)^2}}
\end{aligned} \tag{77}$$

Here $z = (q/q_R)^2$ and $q_R = 0.1973\text{GeV}/c$ corresponding to 1 fermi. The γ's are real constants (but they can be negative) and may be given the normalization

$$\sum_i \gamma_{\alpha i}^2 = 1 \tag{78}$$

In practice, it is probably better to avoid this normalization condition by using the parameters

$$g_{\alpha i}^0 = \gamma_{\alpha i}\sqrt{m_\alpha \Gamma_\alpha^0} \tag{79}$$

as variables in the fit. The residue function is then given by

$$g_{\alpha i}(m) = g_{\alpha i}^0 B_{\alpha i}^l(q, q_\alpha)\sqrt{\rho_i} \tag{80}$$

We define a K-matrix total width $\tilde{\Gamma}_\alpha$ and the K-matrix partial widths $\tilde{\Gamma}_{\alpha i}$ by

$$\tilde{\Gamma}_\alpha = \sum_i \tilde{\Gamma}_{\alpha i} = \Gamma_\alpha(m_\alpha) = \Gamma_\alpha^0 \sum_i \gamma_{\alpha i}^2 \; \rho_i(m_\alpha) \tag{81}$$

From these one finds

$$\begin{aligned} \Gamma_\alpha^0 &= \sum_i \frac{\tilde{\Gamma}_{\alpha i}}{\rho_i(m_\alpha)} \\ \gamma_{\alpha i}^2 &= \frac{\tilde{\Gamma}_{\alpha i}}{\Gamma_\alpha^0 \rho_i(m_\alpha)} \\ g_{\alpha i}^0 &= \sqrt{\frac{m_\alpha \tilde{\Gamma}_{\alpha i}}{\rho_i(m_\alpha)}} \end{aligned} \tag{82}$$

Note that the K-matrix width do not need to be identical with the width which is observed in an experimental mass distribution nor with the width of the $\widehat{T}$-matrix pole in the complex energy plane. We will will refer the former as Γ_{obs}, the latter as Γ_{pole} In the limit in which the masses of the decay particles can be neglected compared to m_α, one has $\Gamma_\alpha(m_\alpha) \simeq \Gamma_\alpha^0$. In terms of the γ's and g^0's, the invariant K-matrix now has a simpler form

$$\begin{aligned} \widehat{K}_{ij} &= \sum_\alpha \frac{\gamma_{\alpha i}\gamma_{\alpha j} m_\alpha \Gamma_\alpha^0 B_{\alpha i}^l(q, q_\alpha) B_{\alpha j}^l(q, q_\alpha)}{m_\alpha^2 - m^2} \\ &= \sum_\alpha \frac{g_{\alpha i}^0 g_{\alpha j}^0 B_{\alpha i}^l(q, q_\alpha) B_{\alpha j}^l(q, q_\alpha)}{m_\alpha^2 - m^2} \end{aligned} \tag{83}$$

Here one allows for the possibility that γ's and g^0's can be negative.

It is possible to parametrize a non-resonant background in each K-matrix element by adding a real unitless constant to the sum of pole terms (83), i.e.

$$\widehat{K}_{ij} \to \widehat{K}_{ij} + c_{ij} \tag{84}$$

Note that unitarity is preserved in this process.

Consider now an isovector P-wave $\pi\pi$ scattering at or near the ϱ mass. Then the elastic scattering amplitude at the $\pi\pi$-mass m is given by

$$K = \frac{m_0\Gamma(m)}{m_0^2 - m^2} = \tan\delta \tag{85}$$

where m_0 is the mass of the ϱ and δ is the usual phase shift. The mass-dependent width is given by

$$\Gamma(m) = \tilde{\Gamma}_0 \left(\frac{\rho}{\rho_0}\right) \left[B^1(q, q_0)\right]^2 \tag{86}$$

where $\tilde{\Gamma}_0$ is the K-matrix width and q (q_0) is the $\pi\pi$ break-up momentum for the $\pi\pi$ mass m (m_0). Neglecting the angular dependence of the amplitude, one obtains

$$T = e^{i\delta}\sin\delta = \left[\frac{m_0\tilde{\Gamma}_0}{m_0^2 - m^2 - im_0\Gamma(m)}\right] [B^1(q, q_0)]^2 \left(\frac{\rho}{\rho_0}\right) \tag{87}$$

The first bracket in (87) contains the usual Breit-Wigner form and the last bracket expresses the two-body phase-space factor. In this simple case, the K-matrix width and the observed width are identical. Note that the phase-space factor is absent in the Lorentz-invariant amplitude $\widehat{T}$ given by (30). The q^2 dependence of the amplitude (in $[B^1(q, q_0)]^2$) reflects the fact that both the initial and the final $\pi\pi$ systems are in P-wave. The normalization for the transition amplitude has been chosen such that

$$T = +i \quad \text{and} \quad \widehat{T} = \frac{+i}{\rho} \quad \text{at} \quad m = m_0 \tag{88}$$

It is seen that the invariant amplitude $\widehat{T}$ is not normalized to 1 but to ρ^{-1}. It is for this reason that the Argand diagram is usually plotted with T and not $\widehat{T}$.

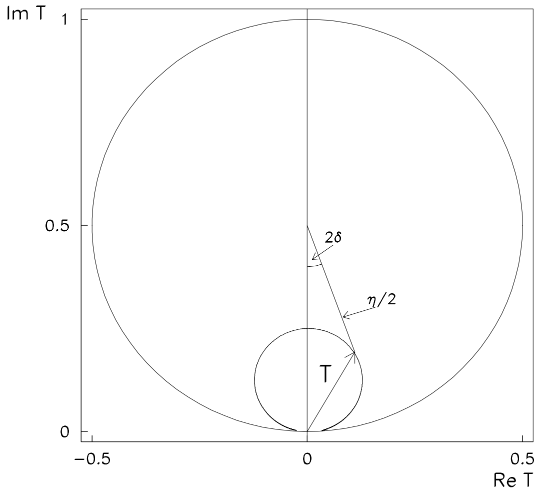

Figure 1:
Argand-plot and definition of the parameters δ and η in case of a ratio of K-matrix widths $\tilde{\Gamma}_2/\tilde{\Gamma}_1 = 0.25$.

In the next example we investigate the influence of coupling of a resonance to a second open channel at the T-matrix parameters (phase-shift δ and inelasticity η) as defined in figure 1. Consider a S-wave resonance at 1500 MeV decaying into $\pi\pi$ and $K\bar{K}$ with a total width of 120 MeV. In the first example we haven choosen the K-matrix widths to be $\tilde{\Gamma}_{\pi\pi} = 30$ MeV and $\tilde{\Gamma}_{K\bar{K}} = 90$ MeV,in the second one we choose $\tilde{\Gamma}_{\pi\pi} = 90$ MeV and $\tilde{\Gamma}_{K\bar{K}} = 30$ MeV. The change of the coupling to the channnel $K\bar{K}$ has no influence on the inelaticity η and on the line shape of $|T|^2$. The only visible difference is the behaviour of the phase motion δ (see figure 2). In the case of a strong coupling to the $K\bar{K}$ channnel ($\tilde{\Gamma}_{K\bar{K}} = 90$ MeV) one cannot decide whether there is a resonance or not by measuring the first channel only (here $\pi\pi$), if the errors of the phase-shift are large.

Consider again a $\pi\pi$ scattering at mass m. But suppose there exist two resonances with masses m_a and m_b coupling to the isoscalar D-wave channel. The prescription for the K-matrix in this case is that

$$K = \frac{m_a\Gamma_a(m)}{m_a^2 - m^2} + \frac{m_b\Gamma_b(m)}{m_b^2 - m^2} \tag{89}$$

i.e. the resonances are summed in the K-matrix. The mass-dependent widths are given by

$$\Gamma_\alpha(m) = \tilde{\Gamma}^0_\alpha \left(\frac{m_a}{m}\right) \left(\frac{q}{q_\alpha}\right) [B^2(q, q_\alpha)]^2 \tag{90}$$

where $\alpha = a$ or $\alpha = b$. q_α is the $\pi\pi$ breakup momentum at $m = m_\alpha$.

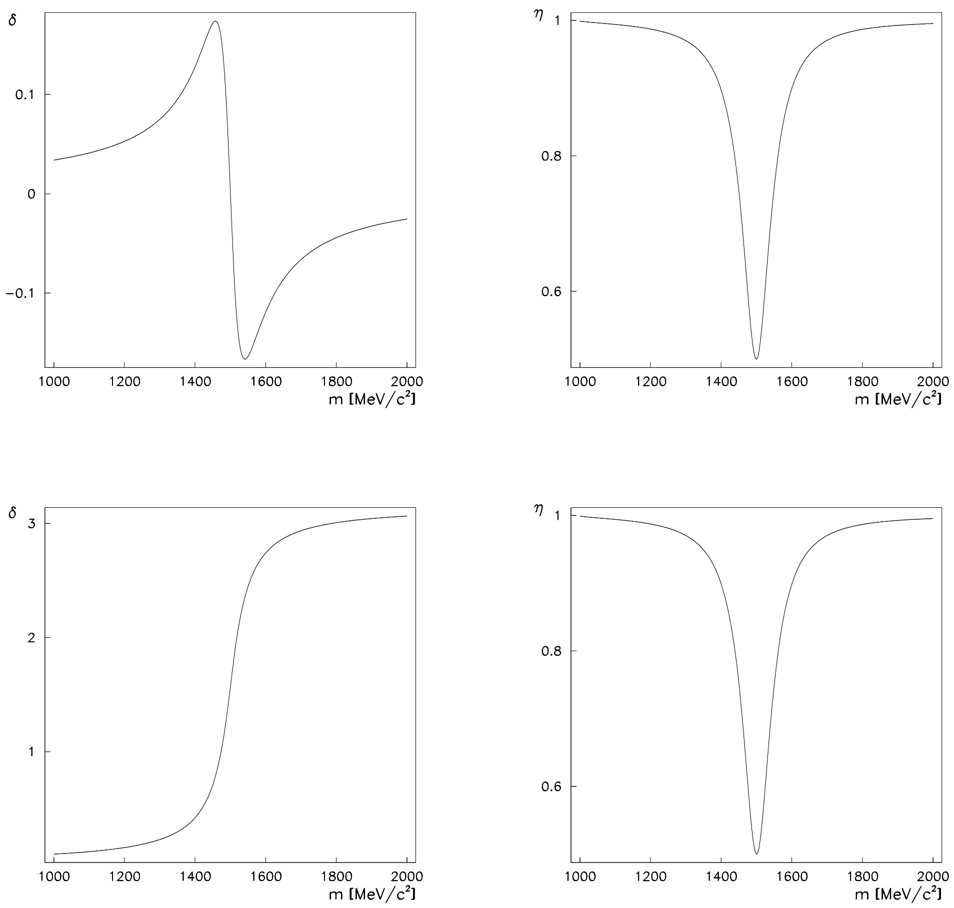

Figure 2: *Phase-shift δ and inelasticity η for a m = 1500 MeV $\tilde{\Gamma}$ = 120 MeV $I = J = 0$ resonance. $\tilde{\Gamma}_{\pi\pi}$ = 30 MeV and $\tilde{\Gamma}_{K\bar{K}}$ = 90 MeV (top) and vice versa (bottom).*

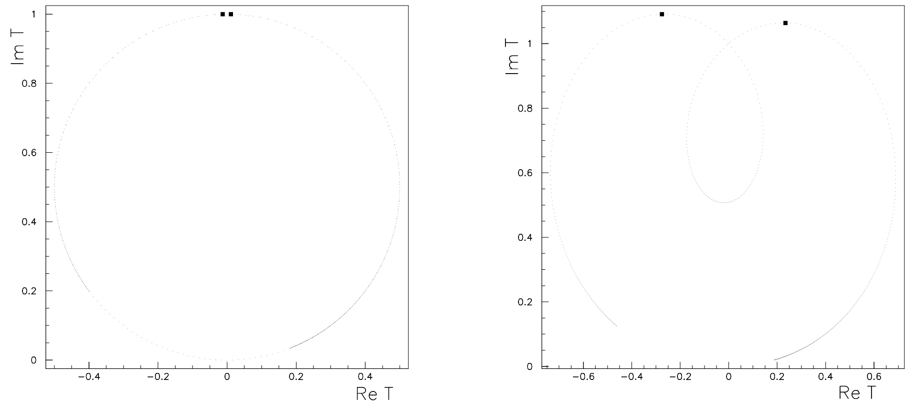

Figure 3: Left : transition amplitude derived from equation (89). Note that it shows one full turn for each resonance. Right : transition amplitude according to (91). The parameters are $m_a = 1.275$ GeV,$\tilde{\Gamma}_a^0 = 0.18$ GeV, $m_b = 1.565$ GeV,$\tilde{\Gamma}_b^0 = 0.16$ GeV; the big dots illustrate the resonance positions.

If m_a and m_b are far apart relative to the widths, then K is dominated either by the first or the second resonance depending on whether m is near m_a or m_b. The transition amplitude is then given merely by the sum

$$\begin{aligned} T &\simeq \left[\frac{m_a\tilde{\Gamma}_a^0}{m_a^2 - m^2 - im_a\Gamma_a(m)}\right]\left[\left(\frac{m_a}{m}\right)\left(\frac{q}{q_a}\right)\right][B^2(q,q_a)]^2 \\ &+ \left[\frac{m_b\tilde{\Gamma}_b^0}{m_b^2 - m^2 - im_b\Gamma_b(m)}\right]\left[\left(\frac{m_b}{m}\right)\left(\frac{q}{q_b}\right)\right][B^2(q,q_b)]^2 \qquad (91) \end{aligned}$$

In the limit in which the two states have the same mass, i.e. $m_c \equiv m_a = m_b$, then the transition amplitude becomes

$$T = \frac{m_c[\Gamma_a(m) + \Gamma_b(m)]}{m_c^2 - m^2 - im_c[\Gamma_a(m) + \Gamma_b(m)]} \qquad (92)$$

This shows that the result is a single Breit-Wigner form but its total width is now the sum of the two individual widths. In case of two nearby resonances (91) is not strictly valid. For a specific example Figure 3 shows the transition amplitude T from the correct equation (89) and from the approximate (91). Note that (91) exceeds the unitary circle.

6 Production of Resonances in P- and Q-vector approach

So far one has considered s-channel resonances, or 'formation' of resonances, observed in the two-body scattering of the type $ab \to cd$. The K-matrix formalism can be generalized to cover the case of 'production' of resonances in more complex reactions. The key assumption is that the two-body system in the final state is an isolated one and that the two particles do not simultaneously interact with the rest of the final state in the production process.

According to Aitchison [15], the production amplitude P should be transformed into F in the presence of two-body final state interactions, as follows:

$$F = (I - iK)^{-1}P = TK^{-1}P \tag{93}$$

Or, taking the invariant form, it may be written

$$\hat{F} = (I - i\widehat{K}\rho)^{-1}\hat{P} = \hat{T}\widehat{K}^{-1}\hat{P} \tag{94}$$

where $\hat{P}$ characterizes production of a resonance and $\hat{F}$ is the resulting invariant amplitude. Note the following relationships:

$$F = \{\rho\}^{\frac{1}{2}}\ \hat{F} \quad \text{and} \quad P = \{\rho\}^{\frac{1}{2}}\ \hat{P} \tag{95}$$

Consider first a single-channel problem, e.g. the isoscalar $\pi\pi$ system in S-wave below 1 GeV. Then, the K is simply given by (24) and one finds

$$\hat{F} = e^{i\delta}\cos\delta\ \hat{P} \tag{96}$$

The final-state interaction brings in a factor $e^{i\delta}$—this is the familiar Watson's theorem [16]. It is emphasized that $\hat{P}$ must have the same poles as those of the K-matrix; otherwise $\hat{F}$ would vanish at the pole position ($\delta = \pi/2$).

In general, $\hat{P}$ and $\hat{F}$ are both column vectors, n-dimensional for an n-channel problem. If the K-matrix is given as a sum of the poles as in (70), then the corresponding

P-vector is

$$P_i = \sum_\alpha \frac{\beta_\alpha^0 \, g_{\alpha i}(m)}{m_\alpha^2 - m^2} \tag{97}$$

and

$$\widehat{P}_i = \sum_\alpha \frac{\beta_\alpha^0 \, g_{\alpha i}(m)}{(m_\alpha^2 - m^2)\sqrt{\rho_i}} \tag{98}$$

where β_α^0 (expressed in units of energy), carries the production information of the resonance α. The constant β_α^0 is in general complex, but it can be set to be real under certain assumptions—this point will be addressed at the end of this section.

The P-vectors should contain the same set of poles as those found in the K-matrix. However, according to Aitchison, one may add a constant term (or a polynomial in energy) to P

$$P_i \to P_i + d_i \tag{99}$$

where the constant d_i is in general complex.

In [17] the production process $\bar{\mathrm{p}}\mathrm{p} \to \pi^+ \ \pi^- \ \pi^0$ was described by a production amplitude for the $\varrho\pi$ and $f_2(1270)\ \pi$ intermediate states. The data required introduction of an additional constant amplitude which was interpreted as that for direct three-pion production. This direct production amplitude can be described in our formalism with the additional constant in the F-vector.

It is often more convenient to rescale β^0's

$$\beta_\alpha^0 = \beta_\alpha \sqrt{m_\alpha \Gamma_\alpha^0} \tag{100}$$

so that β's are unitless. Then the $\widehat{P}$-vectors read, from (79),

$$\widehat{P}_i = \sum_\alpha \frac{\beta_\alpha \gamma_{\alpha i} \, m_\alpha \Gamma_\alpha^0 B_{\alpha i}^l(q, q_\alpha)}{m_\alpha^2 - m^2} \tag{101}$$

where, once again, γ's are real but β's could be complex. If the production process has some known dependance on momentum or angular momentum, the production strength β should be modified accordingly.

It is instructive to work out the above formula in the case of a single resonance in a single channel. Then one has

$$\widehat{P} = \frac{\beta m_0 \Gamma_0}{m_0^2 - m^2} B^l(q, q_0)$$

so that, with $\widehat{K}\rho$ given by (85),

$$\widehat{F}(m) = \beta \frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)} B^l(q, q_0) \tag{102}$$

This is exactly what one writes down for a Breit-Wigner form, except that one has multiplied by an arbitrary constant β and the centrifugal damping factor $B^l(q, q_0)$. This provides a K-matrix justification of the traditional 'isobar' model. Note that the numerator is a constant (independent of m).

The difference of the Breit-Wigner description and the parametrization via the unitarity conserving K-matrix can be demonstrated in the case of two resonances in the same single channel. The formula for the K-matrix is given by equation (89). For example, we consider a production of two nearby resonances in the $\pi\pi$ D-wave, with a production strengh of 2:1. The K-matrix masses and widths are shown in the first column in table 1. The $|\widehat{F}(m)|^2$-distribution is fitted with two Breit-Wigner amplitudes with free masses, widths and production strengths (non-unitarily added). In addition, a phase for the second Breit-Wigner amplitude is left free in the fit. The result is visualized in figure 4. Obviuosly, there are no striking differences between simulation and fit. The parameters of the Breit-Wigner amplitudes are shown in the second column of table 1. Mass and width of the first Breit-Wigner amplitudes are shifted to higher values, the parameters of the second resonance are lowered. So even though the mass distributions are decribed accurately, the masses are reproduced only rather poorly. This difference will be less pronounced in the case of two well separated resonances. This is demonstrated in the second example (see figure 5). The K-matrix parameters are given in the third column of table 1 and the parameters of the Breit-Wigner amplitudes, obtained by fitting the mass distribution, in the fourth column. For well separated resonances in case of a single channel the Breit-Wigner description is obviously an appropriate approximation.

Cahn and Landshoff [6] state that in some approximations the column vector

$$Q = K^{-1} P \quad \text{and} \quad \{\rho\}^{\frac{1}{2}} Q = \widehat{Q} \quad \text{and} \quad \widehat{Q} = \widehat{K}^{-1} \widehat{P} \tag{103}$$

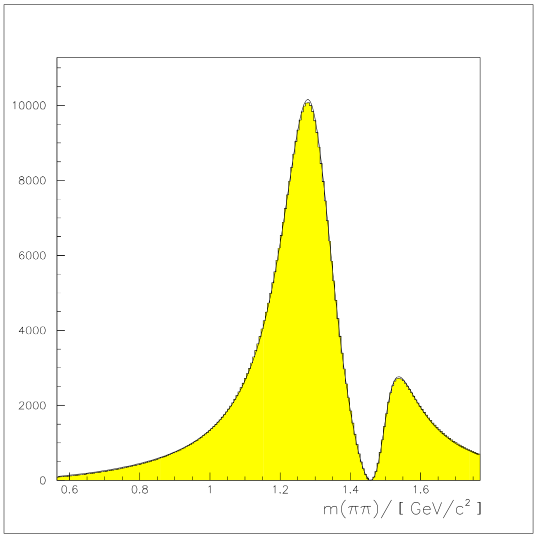

Figure 4:
Two nearby resonances fitted with two Breit-Wigner amplitudes. The shaded area corresponds to the mass distribution generated according to the K-matrix formalism [see (89)]. The solid line represents a fit to the distribution with two Breit-Wigner amplitudes [see (102)].

may be considered a constant in a given limited energy range. Then, one has

$$F = TQ \quad \text{and} \quad \widehat{F} = \widehat{T}\widehat{Q} \tag{104}$$

i.e. the two-body final-state interaction may be expressed as a product of the $\widehat{T}$-matrix and a constant column vector. The $\widehat{Q}$-vector is devoid of the threshold singularities (i.e., no dependence on ρ) and does not contain pole terms. It therefore depends in general on $s = m^2$ only. In a single-channel problem, e.g. the isoscalar $\pi\pi$ system in S-wave below 1 GeV, we now obtain, instead of (96) derived for the P vector approach,

$$\widehat{F} = \frac{1}{\rho} e^{i\delta} \sin\delta \; \widehat{Q} \tag{105}$$

This amplitude contains the familiar scattering amplitude $e^{i\delta}\sin\delta$.

The P- and Q-vector approaches, even though both being approximations for production of multiparticle final states, correspond to different interpretations of the physical processes. For clarity we consider a specific reaction, e.g. $\bar{p}p \rightarrow \pi\pi\eta$ and $\bar{p}p \rightarrow K\bar{K}\eta$ from which we want to extract information on $(\pi\pi)$ interactions. In the Q-vector approach, the amplitude is given by $F_1 = T_{11}Q_1 + T_{12}Q_2$: the $\pi\pi$ system is produced with an amplitude $Q_1(s)$. The two–pion interactions are then taken into account by the scattering amplitude T_{11}. Alternatively, a K and a $\bar{K}$ are produced with amplitudes Q_2 which scatter via T_{12} into the outgoing pions. This picture has to be contrasted to

Parameter	K-matrix	Breit-Wigner	K-matrix	Breit-Wigner
m_1[GeV/c^2]	1.270	1.294	0.9	0.906
Γ_1[GeV/c^2]	0.185	0.197	0.120	0.127
m_2[GeV/c^2]	1.520	1.497	1.560	1.554
Γ_2[GeV/c^2]	0.120	0.113	0.120	0.120
β_1	100.0	114.1	100.0	97.0
β_2	50.0	68.4	100.0	101.0
phase [rad]	0.0	-1.474	0.0	-0.382

Table 1: Table of generated and fitted parameters

that which one may have in mind in applying the P vector approach, $F = (I - iK)^{-1}P$. A particle is produced with an amplitude P, the term $(I - iK)^{-1}$ may be considered as propagator for this particle which then decays.

As an example, one may take the Bowler method [19] for extracting resonance parameters in the presence of the 'Deck' effect. Bowler describes the diffractive production of $\rho\pi$ (or $K^*\pi$ and ρK) with two amplitudes. He proposed to modify the bare Deck amplitude by a factor $e^{i\delta}\cos\delta$ which we have seen in (96), i.e. the P-vector approach. Since the Deck amplitude is non-resonant, this part contains no pole. He adds to this Deck amplitude a $\rho\pi$ scattering amplitude, $e^{i\delta}\sin\delta$, see (105),i.e. the Q-vector approach. Hence the Bowler method contains both P- and Q-vector approaches in one amplitude.

We now turn to a discussion of the contraints one may impose on the P- and Q-vectors, following the argument as outlined by T. L. Trueman (private communication). Consider for this purpose a production of hadrons in the reaction

$$\gamma\gamma \to \text{hadrons} \tag{106}$$

and let F_i be the production amplitude, where the subscript i denotes a particular final hadronic state in the reaction above, i.e.

$$F_i = {}_+\langle i|0\rangle_- \tag{107}$$

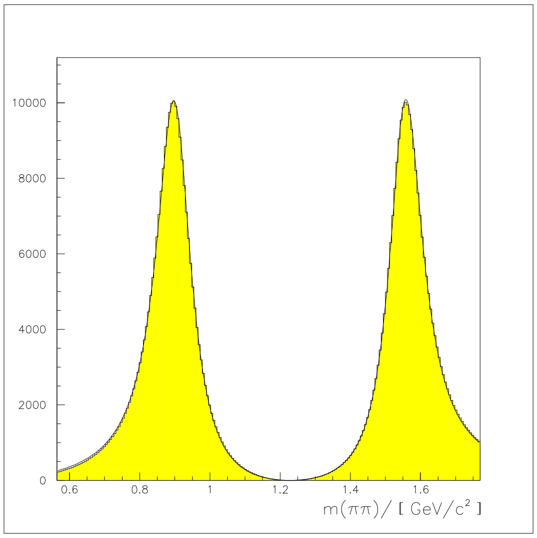

Figure 5:
Two separated resonances fitted with two Breit-Wigner amplitudes. The shaded area corresponds to the mass distribution generated according to the K-matrix formalism [see (89)]. The solid line represents a fit to the distribution with two Breit-Wigner amplitudes [see (102)].

The subscripts $+$ and $-$ here stand for the outgoing and the incoming waves. The ket state $|0\rangle$ stands for the $\gamma\gamma$ initial state. It is to be understood that both $|0\rangle$ and $|i\rangle$ refer to the ket states of a definite spin-parity state [see (4)].

The S-matrix of (32) can be used to connect the incoming and outgoing waves,

$$|i\rangle_- = \sum_k S_{ik}|k\rangle_+ \tag{108}$$

from which it follows that

$${}_+\langle i|j\rangle_- = {}_+\langle i|S|j\rangle_+ = S_{ij} \tag{109}$$

or, equivalently,

$${}_-\langle i|j\rangle_+ = {}_+\langle i|S^\dagger|j\rangle_+ = S^\dagger_{ij} \tag{110}$$

Note that the elements of the S-matrix are defined in terms of a complete set of outgoing waves.

Observe that, with the aid of the anti-unitary time-reversal operator A,

$$\begin{aligned} F_i^* &= {}_+\langle i|0\rangle_-^* \\ &= {}_+\langle i|A^\dagger A|0\rangle_-^* \end{aligned}$$

$$\begin{aligned} &= \left({}_{+}\langle i|A^{\dagger})(A|0\rangle_{-} \right) \\ &= {}_{-}\langle i|0\rangle_{+} \end{aligned} \tag{111}$$

where one has used the relation $A^{\dagger}A = -I$. It should be noted that the time reversal-operator changes the incoming into outgoing wave and vice versa. Applying the closure relationship, one gets

$$\begin{aligned} F_i^* &= \sum_k {}_{-}\langle i|k\rangle_{+}\ {}_{+}\langle k|0\rangle_{+} \\ &= \sum_k S_{ik}^{\dagger}\ {}_{+}\langle k|0\rangle_{+} \\ &= \sum_k S_{ik}^{\dagger} F_k \end{aligned} \tag{112}$$

The last step above involves one crucial step. The ket state $|0\rangle$ contains two γ's and therefore does not undergo strong interaction—as a consequence, one may write

$$|0\rangle_{+} = |0\rangle_{-} \tag{113}$$

In another words, the incoming and outgoing waves do not give rise to 'phase shifts.' From (7), one finds readily that

$$Im\ F = T^{\dagger}F \quad \text{and} \quad Im\ \widehat{F} = \widehat{T}^{\dagger}\rho\widehat{F} \tag{114}$$

which may be compared to (19) and (35).

The above relationship leads to an important result that the Q-vector in $F = TQ$ is real. This fact is readily seen from the following exercise (D. Morgan, private communication). Since both T and T^{-1} are symmetric matrices, one finds

$$\begin{aligned} 2i\, Im\ Q &= T^{-1}F - (T^{-1})^{*}F^{*} \\ &= [T^{-1} - (T^{-1})^{\dagger}]F + (T^{-1})^{\dagger}(F - F^{*}) \\ &= [T^{-1} - (T^{\dagger})^{-1}]F + (T^{\dagger})^{-1}(F - F^{*}) \\ &= 0 \end{aligned} \tag{115}$$

Here one has used both (10) and (114). Since $Q = K^{-1}P$, one concludes that the P-vector must be real as well.

It should be emphasized that the reality condition on the P- and Q-vectors requires three ingredients: (1) the initial state is non-strongly interacting; (2) all available channels are included in T (the closure relationship), i.e. the dimensionality of the T-matrix should be maximal; and (3) the T-matrix is unitary [see (10)].

Consider now an n-particle system produced in an arbitrary production process. One wishes to apply final-state interaction to a given two-body channel out of the n-body system. According to Cahn and Landshoff [6], the P- and Q-vectors should be free of the thresholds in that two-body subenergy under consideration. If one makes a simplifying assumption that these vectors are, in addition, independent of the thresholds of all the other possible combinations of subenergy, then they are real. This 'reality condition' is violated, for example, if one of the particles in the two-body channel under consideration happens to simultaneously interact (via rescattering) with another particle in the n-body system. Finally, one must keep in mind that the reality condition depends on the assumption that the initial system, which could include time-reversed final particles, does not undergo strong interactions.

This 'approximation' has been used by Au, Morgan and Pennington [7], and also in recent works by Morgan and Pennington [20], but in a limited mass range, m $\leq 1.1\mathrm{GeV}/c^2$. The authors argue ([21]) that the isobar model ensures that the two–body unitarity can be applied to the subsystem, and that the Q should remain real. Donnachie and Clegg [22] have used the same assumption in their Breit-Wigner description for the ρ' and ω' analyses. However, Bowler, Game, Aitchison and Dainton [23] used the P-vector given by

$$P_i = D_i + \sum_k \frac{\beta_{ik}\gamma_{ik}}{m_k^2 - m^2} \tag{116}$$

and allow the Deck term D_i to have an arbitrary phase, while keeping the β's and γ's real. Note that this prescription necessarily makes the Q-vector complex. Finally, it should be pointed out that Longacre [18] has consistently used complex β's to characterize the production of resonances. This approach, however, has been criticized by Morgan and Pennington[20].

To the extent that the reality assumption is just that–an approximation– one must regard it as an experimental question and allow complex β's—and explore if the data could accomodate non-zero phases.

7 Simple Examples

As a concrete example one may consider two resonances coupling to two different channels. Consider then two isoscalar $J^{PC} = 0^{++}$ resonances m_a and m_b, with both masses around 1.0 GeV, coupling to $\pi\pi$ (channel 1) and $K\bar{K}$ (channel 2). The elements of the $\widehat{K}$-matrix are, assuming factorization for the residues,

$$\begin{aligned}
\widehat{K}_{11} &= \gamma_{a1}^2 \widehat{\Omega}_a(m) B_{a1}^l(q, q_a) + \gamma_{b1}^2 \widehat{\Omega}_b(m) B_{b1}^l(q, q_b) \\
\widehat{K}_{22} &= \gamma_{a2}^2 \widehat{\Omega}_a(m) B_{a2}^l(q, q_a) + \gamma_{b2}^2 \widehat{\Omega}_b(m) B_{b2}^l(q, q_b) \\
\widehat{K}_{12} &= \widehat{K}_{21} = \gamma_{a1}\gamma_{a2} \widehat{\Omega}_a(m) B_{a1}^l(q, q_a) B_{a2}^l(q, q_a) + \gamma_{b1}\gamma_{b2} \widehat{\Omega}_b(m) B_{b1}^l(q, q_a) B_{b2}^l(q, q_a)
\end{aligned} \tag{117}$$

where

$$\widehat{\Omega}_a(m) = \frac{m_a \Gamma_a^0}{m_a^2 - m^2}, \quad \widehat{\Omega}_b(m) = \frac{m_b \Gamma_b^0}{m_b^2 - m^2} \tag{118}$$

We kept the index l for further use of these relations. Note that Γ_a^0 and Γ_b^0 are constants for scalar resonances coupling to two spinless particles. The normalizations are given by

$$\begin{aligned}
\gamma_{a1}^2 + \gamma_{a2}^2 &= 1 \\
\gamma_{b1}^2 + \gamma_{b2}^2 &= 1
\end{aligned} \tag{119}$$

The $\widehat{P}$-vector can be written

$$\widehat{P} = \begin{pmatrix} \beta_a \gamma_{a1} \widehat{\Omega}_a(m) B_{a1}^0(q, q_a) + \beta_b \gamma_{b1} \widehat{\Omega}_b(m) B_{b1}^0(q, q_b) \\ \beta_a \gamma_{a2} \widehat{\Omega}_a(m) B_{a2}^0(q, q_a) + \beta_b \gamma_{b2} \widehat{\Omega}_b(m) B_{b2}^0(q, q_b) \end{pmatrix} \tag{120}$$

where β_a and β_b are unitless constants specifying production of the resonances m_a and m_b.

The production amplitudes for two resonances in a two-channel problem are obtained by substituting (117) to (120) into

$$\widehat{F}_1 = \frac{\widehat{P}_1 - i\rho_2(\widehat{K}_{22}\widehat{P}_1 - \widehat{K}_{12}\widehat{P}_2)}{1 - \rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \tag{121}$$

$$\widehat{F}_2 = \frac{\widehat{P}_2 - i\rho_1(\widehat{K}_{11}\widehat{P}_2 - \widehat{K}_{12}\widehat{P}_1)}{1 - \rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \tag{122}$$

where $\widehat{D}$ is as given in (51). See Longacre [18] examples of the use of the $\widehat{P}$-vector approach.

For a two-channel problem, one obtains

$$\widehat{F}_1 = \frac{(\widehat{K}_{11} - i\rho_2\widehat{D})\widehat{Q}_1 + \widehat{K}_{12}\widehat{Q}_2}{1 - \rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \tag{123}$$

$$\widehat{F}_2 = \frac{(\widehat{K}_{22} - i\rho_1\widehat{D})\widehat{Q}_2 + \widehat{K}_{12}\widehat{Q}_1}{1 - \rho_1\rho_2\widehat{D} - i(\rho_1\widehat{K}_{11} + \rho_2\widehat{K}_{22})} \tag{124}$$

In the $\widehat{Q}$-vector approach we may assume

$$\widehat{Q}_i = \sum_{j=k}^{n} \alpha_{ij}\ s^j \tag{125}$$

In a linear approximation we will get

$$\begin{aligned} \widehat{Q}_1 &= \alpha_{11} + \alpha_{12}s \\ \widehat{Q}_2 &= \alpha_{21} + \alpha_{22}s \end{aligned} \tag{126}$$

and the amplitudes $\widehat{F}_i$ can be cast into the form

$$\begin{aligned} \widehat{F}_1 &= \widehat{T}_{11}\left(\alpha_{11} + \alpha_{12}m^2\right) + \widehat{T}_{12}\left(\alpha_{21} + \alpha_{22}m^2\right) \\ \widehat{F}_2 &= \widehat{T}_{21}\left(\alpha_{11} + \alpha_{12}m^2\right) + \widehat{T}_{22}\left(\alpha_{21} + \alpha_{22}m^2\right) \end{aligned} \tag{127}$$

Au, Morgan and Pennington [7] used this method with the $\widehat{Q}$-vector expanded as polynomials in $s - 4m_K^2$ in their analysis of the $\pi\pi$ S-wave from double-Pomeron data, in combination with a complicate K-matrix parametrization.

8 Flatté Formula

As a next example we take the isovector S-wave scattering with the $a_0(980)$ coupling to the $\pi\eta$ (channel 1) and $K\bar{K}$ (channel 2) final states. Then the elements of the invariant K-matrix are

$$\begin{aligned}\widehat{K}_{11} &= \frac{\gamma_1^2 m_0\Gamma_0}{m_0^2-m^2}\\ \widehat{K}_{22} &= \frac{\gamma_2^2 m_0\Gamma_0}{m_0^2-m^2}\\ \widehat{K}_{12} &= \widehat{K}_{21} = \frac{\gamma_1\gamma_2 m_0\Gamma_0}{m_0^2-m^2}\end{aligned} \tag{128}$$

The 'reduced' widths are denoted by γ_1^2 and γ_2^2, which are both unitless and satisfy

$$\gamma_1^2+\gamma_2^2=1 \tag{129}$$

Then the $\widehat{T}$-matrix (50) is given as

$$\widehat{T}=\frac{m_0\Gamma_0}{m_0^2-m^2-im_0\Gamma_0(\rho_1\gamma_1^2+\rho_2\gamma_2^2)}\begin{pmatrix}\gamma_1^2 & \gamma_1\gamma_2\\ \gamma_1\gamma_2 & \gamma_2^2\end{pmatrix} \tag{130}$$

If one sets

$$g_i=\gamma_i\sqrt{m_0\Gamma_0} \tag{131}$$

so that

$$g_1^2+g_2^2=m_0\Gamma_0 \tag{132}$$

then

$$\widehat{T}=\frac{\begin{pmatrix}g_1^2 & g_1g_2\\ g_1g_2 & g_2^2\end{pmatrix}}{m_0^2-m^2-i(\rho_1g_1^2+\rho_2g_2^2)} \tag{133}$$

This is the Flatté formula.

The $a_0(980)$ appears as a 'regular' resonance in the $\pi\eta$ system (channel 1). The comparable Breit-Wigner denominator, for m near m_c, is

$$m_c^2-m^2-im_c\Gamma_c$$

in the resonance approximation. One finds, therefore,

$$
\begin{aligned}
m_0^2 &= m_c^2 + \left(\frac{\gamma_2}{\gamma_1}\right)^2 \left[\frac{|\rho_2(m_c)|}{\rho_1(m_c)}\right] m_c \Gamma_c \\
\Gamma_0 &= \frac{m_c \Gamma_c}{m_0 \rho_1(m_c) \gamma_1^2}
\end{aligned}
\tag{134}
$$

in terms of the mass m_c and width Γ_c. Note that ρ_i's have been evaluated at $m = m_c$ where $\widehat{T}$ is expected to attain its maximum value. The above formulas give merely a good starting point; in practice one must vary m_0 and Γ_0 to fit the $\pi\eta$ spectrum. The ratio $(\gamma_2/\gamma_1)^2$ is an unknown (commonly fixed at the $SU(3)$ value of 1.5), but the shape of the square of the amplitudes depends only weakly on this value. Once the ratio is fixed, then γ_1^2 and γ_2^2 are fixed through the normalization condition (129).

Assuming $m_c = 985$ MeV, $\Gamma_c = 80$ MeV and $(\gamma_2/\gamma_1)^2 = 1.5$ one can calculate from (134) a K-matrix mass $m_0 = 998.2$ MeV and width $\Gamma_0 = 301.4$ MeV. These values yield the K-matrix partial widths $\tilde{\Gamma}_{\pi\eta} = 80$ MeV and $\tilde{\Gamma}_{K\bar{K}} = 13.7$ MeV. Figure 6 shows the mass projection derived from (130) ($|\rho_1(m)\widehat{T}_{11}|^2$) by inserting the K-matrix parameters (m_0, Γ_0).

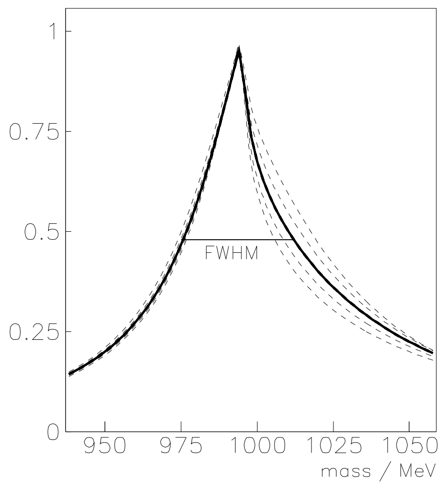

Figure 6:
Simulated mass distribution in the a_0 region using Flatté formula (130). The dashed lines correspond to different ratios of $\alpha = \gamma_2^2/\gamma_1^2$. The increase of FWHM is associated with decreasing values of the ratio α.

The FWHM of this spectrum is approximately 40 MeV. The dashed lines show fits to that mass spectrum but with different fixed ratios $(\gamma_2/\gamma_1)^2$: increasing FWHM corresponds to ratios lower than 1.5 (1.25 and 1.), decreasing FWHM corresponds to ratios higher than 1.5 (1.75, 2.). In fig. 7 the Argand-plot for $\rho_1 \widehat{T}_{11}$ and $\sqrt{\rho_1 \rho_2}\widehat{T}_{12}$ is shown for the a_0. Fig. 8 on the left shows its phase–shift δ_1 together with ϕ_{12}. On the right side of fig. 8 the elasticity η_1 is displayed.

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Figure 7:
Argand plot $\rho_1 \widehat{T}_{11}$ for the simulated a_0. The small loop corresponds to $\sqrt{\rho_1 \rho_2}\widehat{T}_{12}$. The unitary circle is represented by the dashed line.

It is illuminating to explore the pole structure in the Flatté formula, as were done by Morgan and Pennington[12]. Let q_1 and q_2 be the breakup momenta for the $\pi\eta$ and the $K\bar{K}$ channels. The sheet structure in the complex $\sqrt{s} = m$ plane is defined by the signs of (Imq_1, Imq_2): by convention the sheets I through IV are fixed by $(+,+)$, $(-,+)$, $(-,-)$, and $(+,-)$. Consider the energy plane m near $2m_K$. Then one may set approximately $\rho_1 \approx 1$ and $q_2 \ll m_K$. Under this approximation one finds

$$2m_K\, q_1 \simeq 2m_K^2 + q_2^2 \tag{135}$$

If one sets $q_2 = r\, e^{i\phi}$, then one has

$$\mathrm{Im} q_1 \simeq \left(\frac{r^2}{2m_K}\right) \sin 2\phi \tag{136}$$

From this it follows that the sheets for this case may be defined through:
$(\mathrm{Re}\; q_2, \mathrm{Im}\; q_2) = (+,+)$ for sheet I, $(-,+)$ for sheet II, $(+,-)$ for sheet III, and $(-,-)$

for sheet IV.

The denominator of the Flatté formula (133) is now quadratic in q_2 and the two complex roots q_a and q_b may given the expressions, following Morgan and Pennington,

$$\begin{aligned} q_a &= -\alpha + i\beta \\ q_b &= +\alpha - i\gamma \end{aligned} \tag{137}$$

Solving for the roots, one obtains

$$\begin{aligned} g_1^2 &= 4\alpha(\gamma + \beta) \\ g_2^2 &= 4m_K(\gamma - \beta) \\ m_0 &\simeq 2m_K + \frac{\alpha^2 - \beta\gamma}{m_K} \end{aligned} \tag{138}$$

and Γ_0 is given through (132). This shows that α, β, and γ are all positive and furthermore $\gamma > \beta$. Note that the Flatté formula necessarily entails two poles, in sheet II (for q_a) and sheet III (for q_b).

Two roots of q_2 give rise to two poles in m-plane as follows:

$$\begin{aligned} m_a &\simeq 2m_K + \frac{\alpha^2 - \beta^2}{m_K} \\ m_b &\simeq 2m_K + \frac{\alpha^2 - \gamma^2}{m_K} \end{aligned} \tag{139}$$

and

$$\Gamma_a \simeq \frac{4\alpha\beta}{m_K} \quad \text{and} \quad \Gamma_b \simeq \frac{4\alpha\gamma}{m_K} \tag{140}$$

The average mass and width of the two poles are related to m_0 and Γ_0 via

$$\begin{aligned} m_0 &\simeq \frac{m_a + m_b}{2} + \frac{(\gamma - \beta)^2}{2m_K} \\ \Gamma_0 &\simeq \left(\frac{2m_K}{m_0}\right)\left[\frac{\Gamma_a + \Gamma_b}{2} + 2(\gamma - \beta)\right] \end{aligned} \tag{141}$$

where one has used the equations (132) and (138). Note that, if $\gamma-\beta$ is small compared to the widths which are themselves small compared to the masses, then m_0 and Γ_0 are approximately equal to the average mass and width.

With the values of the previous example one obtains for the parameters α = 121.4 MeV, β = 78.6 MeV and γ = 169.3 MeV . From eqn. 139 one calculates $m_a \simeq 1012$ MeV and $m_b \simeq 970$ MeV, from eqn. 140 $\Gamma_1 \simeq 77$ MeV and $\Gamma_b \simeq 165$ MeV.

The Flatté formula provides a simple example of the Jost function representation of the S-matrix[12]. Following Martin et al.[13], one may introduce a 'real analytic' function of q_1 and q_2 with square-root branch points at $q_1 = 0$ and $q_2 = 0$. This function must be real on the real axis of s below the lowest threshold—one way to guarantee this is to require that the function be real in the variables $z_1 = iq_1$ and $z_2 = iq_2$. Note that the denominator of the Flatté formula satisfies this condition:

$$d(q_1, q_2) = m_0^2 - m^2 - i(\rho_1 g_1^2 + \rho_2 g_2^2) \tag{142}$$

with m^2 expressed as a function of q_2, as an example. The above formula can be derived from (69) with the substitution

$$\phi(s) = m_0^2 - m^2 \tag{143}$$

and by using (128) and (131). In the physical region, both q_1 and q_2 are real and positive, so that the function (142) satisfies the 'reality condition'

$$d^*(q_1, q_2) = d(-q_1, -q_2) \tag{144}$$

It is instructive to re-derive the Flatté formula (133) from the S-matrix elements given by (65) and (67)—by simply inserting (142) into (32) and solving for $\widehat{T}$.

9 Extraction of resonance parameters from the T matrix

One should remember that resonance parameters normally quoted are determined from the T matrix defined in the complex energy plane and not from the K matrix. The

T matrix corresponding to a given K matrix is constructed using Equation (16). In general different types of poles may occur, and a complete classification of Riemann poles on all Riemann sheet is beyond the scope of this paper—we refer the reader to the reference [4], for example. Suffice it to state – that in the special case of meson resonances in two channels – resonances show up as poles in $(I - iK)^{-1}K$ on the second and third Riemann sheet.

Using the T matrix formula (130) with K matrix parameters as given in the previous example (i.e. $m_0 = 998.2$ MeV, $\Gamma_0 = 301.4$ MeV and $\gamma_2^2/\gamma_1^2 = 1.5$), the following resonance poles $E^{Sheet} = m_p - i\Gamma_p/2$ in the complex energy plane are found :

$$E^{II} = (969 - i61)\text{MeV} \tag{145}$$
$$E^{III} = (1016 - i68)\text{MeV} \tag{146}$$

Fits to the amplitude squared of our example (see also fig. 6) with different fixed couplings γ_2^2/γ_1^2 are tried. The corresponding pole positions in the complex energy plane are shown in fig. 9. Note that states below the $K\bar{K}$ threshold have their pole on the second sheet, states above this threshold on the third sheet.

As a second example $\pi\eta$ S–wave scattering in one single channel was considered. The existence of two resonances is assumed and parametrized with a two pole K matrix. The K matrix masses and widths are $m_1 = 1300$ MeV , $m_2 = 1400$ MeV and $\tilde{\Gamma}_1 = 80$ MeV and $\tilde{\Gamma}_2 = 200$ MeV respectively. The scattering amplitude squared displayed in fig. 10 shows a strong small dip in the region between the two K matrix poles. The two poles of the corresponding T matrix are found in the unphysical energy plane at $E_1^{II} = (1320 - i14)$ MeV and $E_2^{II} = (1378 - i126)$ MeV (see fig.11). The position of K- and T–matrix poles are marked by lines in fig. 10.

10 Summary

We have presented a description of the K-matrix formalism. The K-matrix is derived from the S-matrix, and hence unitarity in the two-body subsystem is strictly maintained. Relations between S, T and K matrices are given, and scrupulous attention has been given to the precise form of these relations. The Lorentz invariant amplitudes

$\widehat{T}$ and $\widehat{K}$, derived from T and K are also given.

The K-matrix formalism is generalized to arbitrary production processes. Two alternative approaches are presented. In the P vector approach of Aitchinson, the production amplitude is given by $\widehat{F} = (I - i\widehat{K})^{-1}\widehat{P}$. This may be considered as production of a resonance (a pole in $\widehat{P}$) in the primary interaction, and propagation of the resonance before its decay into one of the allowed channels. The second approach was suggested by Cahn and Landshoff. Here, the production amplitude is written in the form $\widehat{F} = \widehat{T}\widehat{Q}$ with $\widehat{Q}$ being a polynomial in s. It corresponds to production (with a spectral amplitude $\widehat{Q}$) of two particles which rescatter as a final–state interaction represented by $\widehat{T}$. Practical examples are given to demonstrate the meaning and significance of these formulae.

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Figure 8: *The left box displays the phase–shift δ_1 together with ϕ_{12}. The phases are identical up to the $K\bar{K}$ threshold. Above the threshold ϕ_{12} falls quickly before rising again while δ_1 continuous its movement up to 180°. In the right box the elasticity η_1 is shown. It drastically drops at the $K\bar{K}$ threshold.*

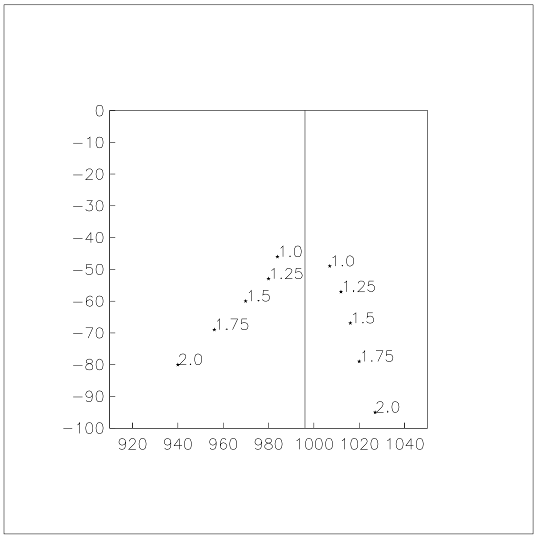

Figure 9:
Pole positions of fits to a simulated a_0 amplitude squared, where different couplings γ_2^2/γ_1^2 are used. Sheet II and sheet III are shown simultaneously, where the line marks the $K\bar{K}$ threshold.

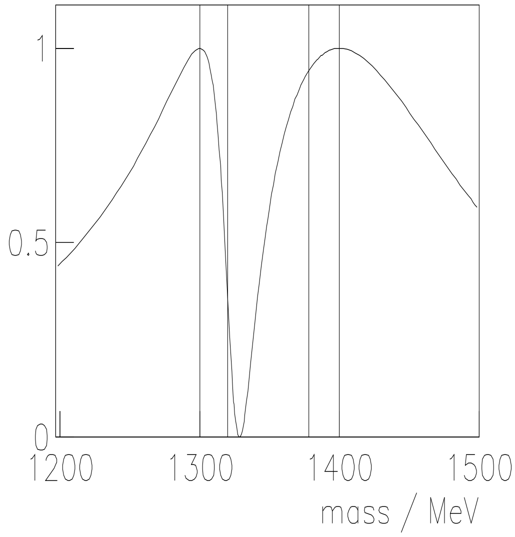

Figure 10:
The amplitude squared of a simulation with two K matrix poles coupling to the $\pi\eta$ channel only. The two inner lines mark the T-matrix mass–pole positions, the outer lines at 1300 MeV and 1400 MeV correspond to the masses used in the K-matrix.

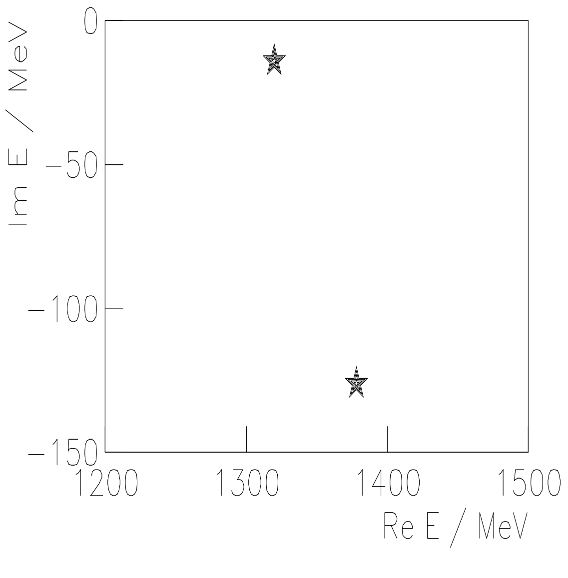

Figure 11:
T–matrix pole positions in the unphysical energy plane for the simulation using K–matrix parameters $m_1 = 1300$ MeV, $m_2 = 1400$ MeV, $\tilde{\Gamma}_1 = 80$ MeV and $\tilde{\Gamma}_2 = 200$ MeV.

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